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# Generalized Differentiation and Summability

For x real let $A_n(x) := a_n \cos nx + b_n \sin nx$ and let

$$(1) \qquad T(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n(x)$$

be a trigonometric series. Suppose that at every $x \in [0,2\pi)$, $T(x) = 0$. Then all $a_n$ and $b_n$ are zero. This is the fundamental theorem in the subject of uniqueness of trigonometric series. It was announced by Riemann in 1854 and the last detail of his proof was supplied in a letter from H.A. Schwarz to Cantor who published it in 1870.[4],[6],[7] The crucial step is this theorem.

**Theorem R.** If F is continuous and

$$RF(x) := \lim_{h \to 0} \frac{F(x-h) - 2F(x) + F(x+h)}{h^2}$$

is zero everywhere, then F is a line.

Theorem R is immediate from a lemma.

**Lemma R.** If F is continuous and $RF \geq 0$ everywhere then F is convex. (See [7],vol.I, p.23, Theorem 10.7.)

Theorem R has only one known proof, namely via Lemma R. To extend

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