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ON A THEOREM OF SHIZUO KAKUTANI

In a paper from 1942 ([1]) Shizuo Kakutani proved the following statement: "let $f(P)$ be a real-valued continuous function defined on a two-sphere S^2. Then there exists a triple of points $P_1, P_2, P_3 \in S^2$ perpendicular to one another, such that $f(P_1) = f(P_2) = f(P_3)$". In the same paper he asks whether the property is still valid when replacing the two-sphere S^2 by a $(n-1)$-sphere S^{n-1} and the 3 points P_1, P_2, P_3 by n points $P_1,\dots,P_n$ $(n \geq 4)$. This problem is still open.

In this paper we prove a plane version of Kakutani's theorem; namely, we replace S^2 by $\bar{\mathbb{C}}$ and the three perpendicular points by the vertices of an equilateral triangle: if f is a continuous mapping of $\bar{\mathbb{C}}$ into $\mathbb{R}$, there exists a triple of points $z_1, z_2, z_3 \in \mathbb{C}$ such that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| > 0$ and $f(z_1) = f(z_2) = f(z_3)$.

The main result we use in proving the nontrivial case of this statement is the following theorem: if $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain, then $\partial\Omega$ contains the vertices of an equilateral triangle. This statement is known if $\partial\Omega$ is a Jordan arc, but an example we give at the end of the proof of the main theorem shows that this weaker form is not sufficient.

We start by proving the result in the strong form.

<u>THEOREM 1</u>: If $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain, then $\partial\Omega$ contains the vertices of an equilateral triangle.

<u>PROOF</u>: We need three lemmas.

<u>LEMMA 1</u>: If $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain, there exist some points $x_1, x_2 \in \partial\Omega$ and $x_3 \in \Omega$ such that $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_1| > 0$.