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DERIVATIVES OF TYPE 1

1. Functions of type k.

The main obstacle in attempts to characterize the class bΔ of bounded derivatives stems from the fact that this class is not closed under outside composition with continuous functions. For example, the following holds (See [1], page 138.):

If $f \in b\Delta$ and $f^2 \in b\Delta$, then f is approximately continuous.

From this result one easily sees that every subclass of bΔ admitting a topological characterization or a characterization in terms of associated sets is contained in the class bA of bounded approximately continuous functions.

There are some bounded derivatives whose properties change after an outside composition with a continuous function in a rather drastic way. Nevertheless, there are also some approximately discontinuous derivatives which behave well even after such a compostion.

Example (See [3], Chapter II, §1, no. 6, exercise 7.):

$$\text{Put} \quad f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

If $\varphi:\mathbb{R} \to \mathbb{R}$ is a continuous function, then the function $\varphi \circ f$ may fail to be a derivative, but it is easy to see that

$$\lim_{s\to 0} s^{-1} \int_0^s \varphi(f(t))dt = \pi^{-1} \int_{(-1,1)} \varphi(y)d(\arc \sin y).$$

Thus the function

$$g(x) = \begin{cases} \varphi(f(x)) & \text{if } x \neq 0 \\ \pi^{-1} \int_{(-1,1)} \varphi(y)d(\arc \sin y) & \text{if } x = 0 \end{cases}$$