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Josef Tkadlec, MFF UK, Sokolovská 83, 186 75 Praha 8, Czechoslovakia

Construction of a finite Borel measure with σ-porous sets as null sets

It is well-known and easy to see that each finite Borel measure on the read line whose null sets contain all sets that are of Lebesgue measure zero as well as of the first category is necessarily absolutely continuous with respect to Lebesgue measure. We show that in this statement one cannot replace the notion of the first categroy sets by the more restrictive notion of σ-porous sets (introduced by Dolženko [1]). Namely, we construct a finite Borel measure μ on the real line such that each σ-porous set is a μ-null set and μ is not absolutely continuous with respect to Lebesgue measure. In the construction we use a special case of a general construction of perfect, non-σ-porous sets given in [2], where also other differences between the class of σ-porous sets and the class of sets of the first category and of Lebesgue measure zero are presented.

For every bounded, open (closed) interval I and for every positive real number c we denote by $c*I$ the open (closed) interval with the same center as I and with length $|c*I| = c.|I|$.

Lemma 1. Let S be a σ-porous subset of the real line and let $c > 1$. Then there is a sequence $\{S_n\}_{n=1}^{\infty}$ of porous sets such that $S = \bigcup_{n=1}^{\infty} S_n$ with the following property for every positive integer n. For every $x \in S_n$ and for every $t > 0$ there exists an open interval $I \subset (x - t, x + t)\setminus S_n$ such that $x \in c*I$.

Proof. It easily follows from [3], theorem 4.5.

Lemma 2. Let μ be a finite Borel measure on a subset S of the real line and let the following conditions hold.

(1) There is $d > 1$ such that $\sum\mu(d*I) < \infty$; the summation being over the set of all bounded intervals I contiguous to $\bar{S}$.