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LEBESGUE POINTS OF FRACTIONAL INTEGRALS

1. Introduction.

Fractional integrals. Let $f \in L(a,b)$ and $\operatorname{re} c > 0$. We define a cth integral of f to be the function $I^c f$ given by

$$I^c f(x) = (I^c f)(x) = \int_a^x \frac{(x-t)^{c-1}}{\Gamma(c)} f(t)\, dt; \tag{1}$$

this is the Riemann-Liouville fractional integral of f of order c.

Much work has been done on integrability- and continuity-type properties of $I^c f$ for various kinds of function f. The main landmark in this is the work of Hardy and Littlewood [1], and it is sometimes thought that they exhausted this field. However, they did not consider Lebesgue points of $I^c f$, and this is the subject of this paper. The main interest is in $0 < c < 1$, to which we confine attention.

A fundamental property is that

$I^c f(x)$ exists for almost all $x \in (a,b)$ and is integrable thereon. (2)

This follows from $I^c f$ being a convolution of integrable functions. However, much more may be true; for instance, considering $c = 1$,

$$I^1 f(x) = \int_a^x f(t)\, dt$$

exists for all $x \in [a,b]$ and is absolutely continuous thereon.

This suggests, and Hardy and Littlewood's many theorems in [1] support, the view that the continuity-type properties of $I^c f$ improve as c increases. For instance, their Theorem 12 shows that under certain conditions $I^c f$ belongs to a Lipschitz class which contracts as c increases. Indeed, the essential message of that theorem amounts, in brief, to: