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SOME REMARKS ON DIFFERENTIAL EQUIVALENCE

The main purpose of this article is to answer some questions arising from the articles of S. Leader [6], [7]. Using the Henstock-Kurzweil integral he introduces a definition of the "differential" df of a function f. This concept is rather broad with not too many properties in general. In order to develop a nice theory a class of functions, called "dampable", is introduced and it is shown that most of the familiar calculus manipulations with differentials can be verified for this class of functions in a satisfying and natural manner. Neither article characterizes this class, and it is our purpose here to give that characterization.

The characterization will be no surprise. Just as for the Lebesgue integral the classes of VB and AC functions arise with compelling regularity, in any study of the Henstock-Kurzweil (alias Denjoy-Perron) integral the classes of VBG_* and ACG_* functions intrude everywhere. Indeed Ward [14] in his study of the Perron-Stieltjes integral, which is intimately related to these matters, suggests that the class of VBG_* functions is the largest class that should arise in these kind of matters.

The key concept needed in presenting this material is the notion of "differential equivalence" (in the language of Kolmogorov [5]) or "variational equivalence" (in the language of Henstock [4]). This is a true equivalence relation and the equivalence classes are what Leader calls his "differentials". In the first section we sketch the apparatus needed for this presentation, in what appears to be a convenient and useful language. Most of the terminology is modelled after standard sources; for example the term "covering relation" is taken from Federer [2]. Proofs here are omitted but may be constructed from the material in [12] and [13]. Section two contains a brief account of the notion of differential equivalence and some basic differentiation results; the proofs of the main results (2.8), (2.9), and (2.10) are given in detail. Finally section three then contains the characterization of dampable functions and its proof.

§1. Notation and preliminaries. Throughout $[a, b] \subset R$ is a fixed interval and all functions are real-valued functions defined on that