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LENGTHS OF RECTIFIABLE CURVES IN 2-SPACE

Let MC denote the family of nondecreasing continuous functions on $[0,1]$, and let BC denote the family of continuous functions of bounded variation on $[0,1]$. Throughout this paper $(g(t), f(t))$ $(0 \leq t \leq 1)$ denotes a continuous rectifiable curve in R^2, i.e., $f,g \in BC$. We propose to determine the length L of this curve in terms of the functions f and g. A well-known result [5, p. 123] is

Proposition 1. We have $L \geq \int_0^1 ((f')^2 + (g')^2)^{1/2}$, and equality holds if and only if f and g are absolutely continuous on $[0,1]$.

We want to express L in terms of f and g in a more general setting. To this end, we introduce a notation from [3]. If A is any subset of $[0,1]$, measurable or not, let

$$M(F,A) = \lim_{m\to\infty} \sum_{i=1}^{2^m} \lambda F(J_{im} \cap A)$$

where λ is Lebesgue outer measure and $J_{im} = [(i-1)2^{-m}, i2^{-m}]$. Note that the expression after the limit increases with m. Moreover, $M(F,A) \leq V(F)$, the total variation of F on $[0,1]$. If F is monotonic, clearly $M(F,A) = \lambda F(A)$. Also $M(F,A) = 0$ if and only if $\lambda F(A) = 0$.

Let $E_F = \{x : F'(x) = \pm\infty\}$. We need the

Definition: We say that f is <u>compatible</u> with g if there exist sets S_f and S_g such that $S_f \cup S_g = E_f \cap E_g$ and $\lambda f(S_f) = \lambda g(S_g) = 0$.

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