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THE LATTICE GENERATED BY DIFFERENTIABLE FUNCTIONS

The purpose of the present paper is to describe the lattice generated by the family of all differentiable functions. It answers a question posed by Z. Grande in [1].

Let us establish some of the terminology to be used. $\mathbb{R}$ denotes the real line. For every function $f : \mathbb{R} \to \mathbb{R}$ $N(f)$ denotes the set of all points at which f is not differentiable.

The symbol $\mathcal{D}$ stands for the family of all differentiable functions. The symbol $\mathcal{R}$ denotes the family of all continuous function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(1) the set $N(f)$ is a finite union of discrete sets,

(2) for every $x \in \mathbb{R}$ the right-hand derivative $f'_+(x)$ and the left-hand derivative $f'_-(x)$ exist at x.

Observe that $\mathcal{D} \subset \mathcal{R}$.

A family $\mathcal{A}$ of real functions is a lattice iff $\max(f,g) \in \mathcal{A}$ and $\min(f,g) \in \mathcal{A}$ for $f,g \in \mathcal{A}$. If $\mathcal{B}$ is a family of real functions, then $\mathcal{L}(\mathcal{B})$ denotes the lattice generated by $\mathcal{B}$, i.e., the smallest lattice of functions containing $\mathcal{B}$.

The following question was posed in [1].

"<u>Problem 7</u>. What is the smallest lattice of functions containing all differentiable functions? Is it the family of all continuous functions differentiable at every point except perhaps at the points of a set which is a finite union of discrete sets?"

In this paper we shall prove that the lattice $\mathcal{L}(\mathcal{D})$ generated by the family of all differentiable functions is equal to the family $\mathcal{R}$.

<u>Theorem 1</u>. The family $\mathcal{R}$ is a lattice of functions.

<u>Proof</u>. Let $f,g \in \mathcal{R}$. We shall prove that $h = \max(f,g)$ belongs to $\mathcal{R}$. First, we shall verify that the set $M = N(h) \setminus (N(f) \cup N(g))$ is discrete and