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THE SINGULARITY OF EXTREMAL MEASURES

0. Introduction.

Let λ be Lebesgue measure on R. A Borel measure μ on $I \times I$ is doubly-stochastic if $\mu(A \times I) = \mu(I \times A) = \lambda(A)$ for each Borel set $A \subseteq I$. The collection of all doubly-stochastic measures forms a convex, weakly compact set whose extreme points have been much studied: [2], [3], [4], [5], [6]. It was shown by Lindenstrauss [5] that every extreme doubly-stochastic measure is singular with respect to planar Lebesgue measure λ^2. It is our purpose to strengthen this result in a general context.

For example, suppose that $L_1,\ldots,L_m$ are lines through the origin in R^2 and that ν is a probability measure on R^2. Then one can consider the convex set of probabilities on R^2 whose projections onto $L_1,\ldots,L_m$ agree with those of ν. Theorem 2.1 infra will say that the extreme points of this set are singular with respect to Lebesgue product measure, no matter what the choice of ν! In the doubly-stochastic case, $m = 2$, L_1 and L_2 are the co-ordinate axes, and ν may be taken as λ^2 restricted to $I \times I$.

1. Preliminary results

A σ-algebra A of subsets of X is countably generated (c.g.)