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Jan Malý, matematicko-fyzikální fakulta UK,
186 00 Praha 8, Sokolovská 83, Czechoslovakia

Perfect level sets in many directions

The so-called locally recurrent functions are defined by the following property: for every $c \in R$ the set $\{x : f(x) = c\}$ is perfect. (We include the empty set among the perfect sets.) The existence of a nonconstant continuous localy recurrent function is not obvious, but there are several known examples. (See e.g. Bush (1962, [2]).) In the present paper we construct a continuous function f on $[0,1]$ such that the functions $f - \lambda\mathrm{id}$ are locally recurrent for every $\lambda \in \Lambda$, where $\Lambda \subset \mathbb{R}$ is a given countable set. (The function id is defined by $\mathrm{id}(x) = x$.) This construction cannot be improved to Λ being uncountable because of the results of Bruckner and Garg (1977, [1]) concerning the level sets of arbitrary continuous functions. (Gillis (1939, [3]) claimed that one can take $\Lambda = \mathbb{R}$, but this is a mistake.) Further, we show that every continuous function on $[0,1]$ can be expressed as the sum of two locally recurrent functions.

__Definition__. A function u on $[0,1]$ is termed __admissible__ if there is a finite set $A_u \subset (0,1)$ such that

(1) if $I \subset [0,1]\setminus A_u$ is an interval, then u is linear on I,

(2) $u(x) < \liminf_{y\to x} u(y)$ for every $x \in A_u$.

__Lemma__. Let s, $-t$ be admissible functions on $[0,1]$, $t \le s$ on $[0,1]$ and $t < s$ except on a finite set. Let $\varepsilon > 0$ be given. Then there are admissible functions s^*, $-t^*$ on $[0,1]$ such that

(1) $t \le t^* \le s^* \le s$ on $[0,1]$,

(2) $t^* < s^*$ except on a finite set,

(3) $s^* - t^* < \varepsilon$ on $[0,1]$,

(4) if f is a continuous function on $[0,1]$, $t^* \le f \le s^*$, then for each $x \in [0,1]$ there is $y \in [0,1]$ such that $0 < |x - y| < \varepsilon$ and $f(x) = f(y)$.