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A CONCEPT OF DIFFERENTIAL BASED ON VARIATIONAL EQUIVALENCE UNDER GENERALIZED RIEMANN INTEGRATION

For appropriate types of integral the variational equivalence $S \sim T$ of two objects of integration S,T is the relation $\int |S-T| = 0$. Kolmogorov [11] introduced such a notion, aptly called "differential equivalence," for set functions. He discussed its basic properties and even noted the differential invariance of Lipschitz functions. Variational equivalence has been used in the development of the generalized Riemann integral [6], [7], [8]. It is essential for the definition of the variational integral. But we contend it has a more important role to play. If $S \sim T$ and S is integrable then so is T and moreover $\int S = \int T$. Thus the ultimate object of integration in not S itself but the equivalence class $\sigma = [S]$ to which S belongs. Our contention here is that these equivalence classes provide a viable mathematical formulation for a concept of differential. Differentials defined in this way greatly facilitate the study of the integral and afford easy access to its applications. We gain a rigorous foundation for a calculus of differentials that includes differentials of discontinuous functions.

In this survey we explore the feasibility of integrational definition of differential by applying it to the exposition of a specific type of integral. We use a modification of Kurzweil's generalized Riemann integral [8]. Where Kurzweil allows the tag for a cell to be any point in the cell we demand that the tag be a vertex of the cell. The differentials induced by this integral have many desirable properities. A suitable subclass of them conforms to the classical formulas of differential calculus. Our differentials yield elegant