Generalizations of bounded variation for $1 \leq p < \infty$ and $k \geq 1$.

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1. Definitions and Notations.

All functions to be considered will be defined on a finite interval [a,b] and may take real or complex values.

We denote by p any real number such that $1 \leq p < \infty$, and by k any positive integer.

If $a \leq x_0 < x_1 < \ldots < x_n \leq b$, the sequence (x_i), $i = 1,2,\ldots,n$, will be called a subdivision of [a,b].

The first divided difference $[F(x_1) - F(x_0)]/(x_1 - x_0)$ is denoted by $Q_1(F; x_0, x_1)$ and the k-th order divided difference over the points $x_0, x_1, \ldots, x_k$ is defined inductively by

$$Q_k(F; x_0, x_1, \ldots, x_k) = (x_k - x_0)^{-1}[Q_{k-1}(F; x_1, \ldots, x_k) - Q_{k-1}(F; x_0, \ldots, x_{k-1})].$$

Then $Q_k(F; x_0, x_1, \ldots, x_k)$ may be written as

$$\sum_{i=0}^{k} \frac{F(x_i)}{\prod_{j \neq i} (x_i - x_j)} ,$$

see [5].

For $h > 0$, the first forward difference $F(x+h) - F(x)$ is denoted by $\Delta_h F(x)$ and, for all $k \geq 2$, $\Delta_h^k F(x)$ is defined inductively by $\Delta_h(\Delta_h^{k-1} F(x))$. It admits the identity

$$\Delta_h^k F(x) = k!\, h^k Q_k(F; x, x+h, \ldots, x+kh) .$$

For any given variation V, the set of functions for which this variation is finite will be