Monotonicity and the Approximate Symmetric Derivative

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In the following, the symmetric derivative of a real-valued function is written as

$$f^s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

The approximate symmetric derivative, $f^{as}(x)$, is defined in the natural way by replacing the ordinary limit with the approximate limit. The relationship between monotonicity and the ordinary symmetric derivative is quite well understood. This relationship is fairly succinctly summed up by the following theorem.

Theorem 1. ([L1]) If $f:\mathbf{R} \to \mathbf{R}$ is such that $f^s \geq 0$ everywhere and $C(f)$ is the set on which f is continuous, then $f|_{C(f)}$ is nondecreasing.

Because every symmetrically differentiable function is differentiable almost everywhere ([U], [K]), Theorem 1 shows that the monotonicity behavior of the symmetric derivative is "almost" that of the ordinary derivative.

In the case of the approximate symmetric derivative, the monotonicity question is still largely open. The most reasonable conjecture seems to be that if $f:\mathbf{R} \to \mathbf{R}$ is measurable and $f^{as} \geq 0$ everywhere, then f is monotone when restricted to

$$A(f)=\{x: f \text{ is approximately continuous at } x\}.$$

There is some circumstantial evidence to support this conjecture.

Theorem 2. ([M]) If f is a measurable function such that $f^{as} \geq 0$ everywhere, then given an interval I, there exists a subinterval J of I such that $f|_{A(f)}$ is nondecreasing on J.

The only other successful attack on the monotonicity question with the approximate symmetric derivative is the following theorem.

Theorem 3. ([P]) Let $f:\mathbf{R} \to \mathbf{R}$ be a Darboux-Baire 1 function and for each $\alpha \in \mathbf{R}$ define $\Lambda_\alpha(f) = \{x : f(x)=\alpha\}$ and

$$K=\{\alpha : \Lambda_\alpha(f) \text{ contains at most countably many approximate maxima of } f\}.$$

If $f^{as} \geq 0$ everywhere and K is dense in $\mathbf{R}$, then f is nondecreasing.