Generalized Riemann Derivatives

by

P.D. Humke and M. Laczkovich

Let a_i, b_i $(i=1,\dots,n)$ be real numbers such that the b_i's are distinct, $\sum_{i=1}^{n} a_i b_i^k = 0$ for $k=0,1,\dots,r-1$ and $\sum_{i=1}^{n} a_i b_i^r = 1$. The generalized Riemann derivative (GRD) of order r of the real function f at the point x is defined by

$$D^r f(x) = \lim_{h\to 0} \frac{\sum_{i=1}^{n} a_i f(x+b_i h)}{h^r/r!} .$$

Replacing the limit by limsup and liminf, we obtain the bilateral upper and lower GRD's $\overline{D}^r f(x)$ and $\underline{D}^r f(x)$, respectively. The unilateral GRD's $\overline{D}^r_+ f(x)$, etc. are defined by restricting h to be positive or negative. Obviously, this notion of derivative depends on the choice of the numbers a_i, b_i. However, it is easy to see that if f has a finite ordinary r^{th} derivative, or, more generally, a finite r^{th} Peano derivative at x then $D^r f(x) = f_{(r)}(x)$, independently of the choice of a_i and b_i. These derivatives, which contain the symmetric and Riemann-Schwarz derivatives as special cases, were investigated by M. Ash in his thesis and