Generalized Riemann Complete Integrals

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Definition. A tagged division of $[a,b]$ will be called a restricted tagged division of $[a,b]$ if it has the form

$$a = x_0 = z_1 < x_1 < z_2 < x_2 < z_3 < \cdots < x_{m-2} < z_{m-1} < x_{m-1} < z_m = x_m = b$$

where $x_0 = z_1$ is the tag of $[x_0,x_1]$, $x_m = z_m$ is the tag of $[x_{m-1},x_m]$ and $z_j$ is the tag of both $[x_{j-1},z_j]$ and $[z_j,x_j]$ for $j = 2,3,\ldots,m-1$. If a restricted tagged division of $[a,b]$ has further the property that $z_j - x_{j-1} = x_j - z_j$, $j = 2,3,\ldots,m-1$, the division will be called a restricted symmetric tagged division of $[a,b]$.

It is clear that given $\delta(x) > 0$ defined on $[a,b]$ there exists a restricted tagged division of $[a,b]$ compatible with $\delta(x)$. That there exists a restricted symmetric tagged division of $[a,b]$ compatible with $\delta(x)$ follows from [2].

If $f$ is a finite function defined on $[a,b]$, let two interval functions be defined by $F_\ell \equiv F_\ell(f,u,v) \equiv f(v)(v-u)$ and $F_r \equiv F_r(f,u,v) \equiv f(u)(v-u)$. It will be convenient to denote a pair of interval functions by a single letter in script face. For example we shall write $\mathcal{F}(u,v) = \{F_\ell(u,v), F_r(u,v)\}$ or, more briefly, $\mathcal{F} = (F_\ell, F_r)$.

Definition. The number $I$ will be called the generalized Riemann complete (generalized symmetric Riemann complete) integral of $f$ with respect to the pair of interval functions $h(u,v) = \{h_\ell(u,v), h_r(u,v)\}$ on $[a,b]$ if, corresponding to $\epsilon > 0$, there is a function $\delta(x) > 0$ so that