Chaotic Behavior and Equicontinuity of Iterates of an Interval Map

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Let f be a continuous function mapping an interval I into itself. For $n=1,2,\ldots$, let $f^{n+1}=f\circ f^n$. Research in the social, biological and physical sciences often leads to a study of the sequence $\{f^n\}$ of iterates [M],[Y],[VSK]. Of importance to the researchers is the question, "if $|x-y|$ is small, will $|f^n(x)-f^n(y)|$ be small for all n?". For example, if x_0 denotes the actual initial population of a species of insect and y_0 the population as estimated by the researcher, one would hope that $|f^n(x_0)-f^n(y_0)|$, the error in estimating the population of the n^{th} generation, would be small if the initial estimate were good. In mathematical language, one would like the family $\{f^n\}$ to be equicontinuous.

Practical problems, however, don't usually lead to equicontinuity of $\{f^n\}$. In fact, one often finds chaotic behavior of various sorts. Theoretically, there could be almost certainly that chaos will arise. Bruckner and Hu [BH] recently established the following result.

Theorem. There exists a continuous function f mapping [0,1] into itself and a set S of Lebesgue measure one such that for $x,y\in S(x\neq y)$,

$$\limsup_{n\to\infty}|f^n(x)-f^n(y)|=1,\quad \liminf_{n\to\infty}|f^n(x)-f^n(y)|=0.$$

Thus with this function f, one can be almost sure that both the true initial value x_0 and the estimate y_0 will be in S, and using $f^n(y_0)$ to predict $f^n(x_0)$ is of no value. The function f can be chosen arbitrarily close (uniformly) to the function $g(x)=4\alpha(1-x)$, a function in the logistic family often arising in practice [M],[P]. We should mention that in practice chaotic situations do arise, but situations in which satisfactory prediction is possible for x_0 and y_0 in some set large in measure or category also arise [P].

The purpose of this article is to determine conditions under which the ideal situation, equicontinuity of the family