# Limits under the integral sign

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Many parts of the calculus need the interchange of limit and integral, for example, the continuity and differentiability of an integral with respect to a parameter, an infinite series of integrals, and the exchange of order of integration in a repeated integral. As we can now define Lebesgue and even Denjoy-Perron integrals by using Riemann sums, it is possible to find necessary and sufficient conditions for the property

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx,$$

given that $f_n(x)$ tends to a finite limit $f(x)$ almost everywhere, and that each $f_n(x)$ is integrable over $[a,b]$.

Assuming generalized Riemann integration theory, the necessary and sufficient conditions are that

(1) $(E)\Sigma f_{m(x)}(x)(v-u)\in C$

for some compact set $C$ of arbitrarily small diameter, some finite positive function $M(x)$ on $[a,b]$, all positive integer valued functions $m(x)\geq M(x)$ on $[a,b]$, some function $\delta(x)>0$ on $[a,b]$, and all $\delta$-fine divisions $E$ of $[a,b)$;

given $\varepsilon>0$, there are a number $F$, an integer $N>0$, and a function $\delta_n(x)>0$ on $[a,b]$ and depending on $n$, with

(2) $F-\varepsilon<(E)\Sigma f_n(x)(v-u)<F+\varepsilon$

for all $\delta_n$-fine divisions $E$ of $[a,b)$ and all $n\geq N$.