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CLASSICAL PROBLEMS IN ANALYSIS AND NEW INTEGRALS

I. The classical approach

We shall motivate and describe recent attempts to define, through the use of suitable limits of Riemann sums, integrals which can integrate the divergence of mere differentiable vector fields. Their approach via nonstandard analysis will be sketched in Part II.

If $\Omega \subset \mathbf{C}$ is an open domain and $f: \Omega \to \mathbf{C}$ is $\mathbf{C}$-differentiable, the Cauchy theorem asserts that

$$\int_{\partial I} f(z)dz = 0 \qquad (1)$$

for each 2-interval I whose closure is contained in Ω. Two types of proof are generally offered in textbooks. The first type uses the Green formula to transform the real and imaginary part of the left-hand side in (1) into integrals on I whose integrands are equal to zero by the Cauchy-Riemann formulas. Such an approach requires more regularity on f than the $\mathbf{C}$-differentiability, for example f continuously $\mathbf{C}$-differentiable. The second type of proof, which goes back to Goursat (1900) [3] proves (1) under the mere $\mathbf{C}$-differentiability assertion on f by a contradiction argument and the technique of successive divisions of I. Such a proof is similar to that of a lemma stated and proved in $\mathbb{R}^2$ in 1895 by Cousin and which has played an important role since the late fifties in the definitions of Perron-type integrals through Riemann sums introduced by Kurzweil [9] and Henstock [4]. To formulate precisely the lemma in $\mathbb{R}^n$, let $I =]a_1, b_1] \times \dots \times]a_n, b_n]$ be a right-closed interval in $\mathbb{R}^n$, $\bar{I}$ its closure and let us call <u>gauge</u> on $\bar{I}$ any positive function defined on $\bar{I}$ <u>L-partition</u> Π of I any finite family

$$\Pi = \{(x^1, I^1), \dots, (x^q, I^q)\}$$

such that the right-closed subintervals I^j of I partition I and the x^j are