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CHARACTERISTIC FUNCTIONS THAT ARE PRODUCTS OF DERIVATIVES

Let D be the system of all (finite) derivatives on the real line R . For each set $A \subset R$ let C_A be its characteristic function. Let \mathcal{G} be the system of all sets $A \subset R$ such that $C_A = fg$ for some $f, g \in D$. (It is not difficult to prove that every closed set belongs to \mathcal{G} .) Since each derivative is a Baire 1 function and since $A = \{x; C_A(x) \geq 1\} = \{x; C_A(x) > 0\}$, we see that every set in \mathcal{G} is ambiguous (i.e. at the same time an F_σ -set and a G_δ -set). Now let $A \subset R$, $f, g \in D$, $C_A = fg$, $p, x_n, y_n \in R$, $p < x_n < y_n$ ($n = 1, 2, \dots$) and $\liminf \frac{y_n - x_n}{y_n - p} > 0$. Let $f = F'$, $g = G'$. It is easy to prove that $\frac{F(y_n) - F(x_n)}{y_n - x_n} \rightarrow F'(p)$ ($= f(p)$); similarly for G . Write $J_n = (x_n, y_n)$ and suppose that $J_n \subset A$ for each n . Using the Cauchy inequality and the Darboux property of derivatives we get $(y_n - x_n)^2 = \left(\int_{J_n} \sqrt{fg}\right)^2 \leq \int_{J_n} f \cdot \int_{J_n} g = (F(y_n) - F(x_n)) \cdot (G(y_n) - G(x_n))$ for each n . Dividing by $(y_n - x_n)^2$ and passing to the limit we obtain $1 \leq f(p) \cdot g(p) = C_A(p)$ so that $p \in A$. Hence: If $A \in \mathcal{G}$, $B = R \setminus A$ and $p \in B$, then such intervals J_n do not exist. (Intuitively: There are no essential holes in B close to p .) This (and a "symmetrical" argument) shows that B is nonporous (i.e. nonporous at p for each $p \in B$). Since A is ambiguous if and only if B is, we have the following simple result: If $A \in \mathcal{G}$, then B is ambiguous and nonporous.

It can be proved that these two properties of B imply that $A \in \mathcal{G}$. Actually, we have a more precise statement:

Theorem 1. Let $A \subset R$, $B = R \setminus A$. Then the following three conditions 1), 2) and 3) are equivalent to each other:

1) There is a natural number m and functions $f_1, \dots, f_m \in D$ such that $C_A = f_1 \cdots f_m$.