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ON PROJECTIONS OF PLANAR SETS

If $a \in R$ and $A \subseteq R^2$, we say that the projection of A in direction a is $\{c : gr(y = ax + c) \cap A \neq \emptyset\}$. Here $gr(y = ax + c)$ denotes the graph of the function $y = ax + c$. The c-projection of A in direction a is $\{c : dom[gr(y = ax + c) \cap A]$ is of second category$\}$. The m-projection (m^*-projection) of A in direction a is $\{c : m(dom[gr(y = ax + c) \cap A]) > 0\}$ ($\{c : m^*(dom[gr(y = ax + c) \cap A]) > 0\}$). Here m denotes Lebesgue measure in R and m^* denotes outer Lebesgue measure in R. The following question was formulated in [1]: "Does there exist a linear set A of second category such that the projection of $A \times A$ onto each line has empty interior?" The partial solution if Martin's Axiom is assumed was provided in [7]. The solution under CH was given in [2] by Davies. Observe that the proof of the theorem of Davies does not really require CH (change "$< \omega_1$" to "τ", "only countable" to "fewer than τ" and recall that uncountable closed sets must have cardinality τ). Thus we obtain Proposition 1.

Proposition 1. There exists a second category set A such that the projection of $A \times A$ in each direction does not contain an interval.

In Proposition 2 we construct a c-Lusin set L for which every c-projection of $L \times L$ in direction $a \neq 0$ is equal to R (under MA). If L is of the first category, then, as is well known, any c-projection of $L \times L$ is empty. In Proposition 3 we construct a strong, first category set S for which every m^*-projection of $S \times S$ in direction $a \neq 0$ is equal to R (under MA). Let $C \subseteq R - \{0\}$ be a set of cardinality less than that of the continuum. In Proposition 4 we construct sets A and B of the second category such that every projection of $A \times B$ in direction $c \in C$ equals R and every c-projection of $A \times B$ in direction $c \in C$ is empty (under MA). We use a technique due to Erdös, Kunen and Mauldin [3].

We use the following notation. If $A,B \subseteq R$, then $A+B = \{a+b : a \in A, b \in B\}$, $A \cdot B = \{ab : a \in A, b \in B\}$, $A-B = \{a-b : a \in A, b \in B\}$ and $A \setminus B =$