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Infinite Peano derivatives

Recall that a function $f:\mathbb{R}\ \mathbb{R}$ has a (finite) n th Peano derivative at x means that there are numbers $f\ (x)$, $f\ (x)$, $\cdots {}_n f\ (x)$ such that

(1) $f(x + h) = f(x) + hf\ (x) + \cdots + h^n f(x)/n! + o(h^n)$ as $h\ \ 0$.

If (1) holds as $h\ \ 0^+$, then we say that f has an n th Peano derivative from the right at x and denote the numbers instead by $f_+\ (x), \cdots, f_{n+}\ (x)$.

If f has an $(n - 1)$ th Peano derivative at x and if

$$(2) \qquad \lim_{h} \frac{f(x + h) - f(x) - \cdots - h^{n-} f_{n-}(x)/(n-1)!}{h^n/n!} = + ,$$

then we write $f_n(x) = +$. We define $f_n(x) = -$ in a similar way. Furthermore $f_{n+}(x) = +$ or $-$ is defined by letting $h\ \ 0$ in$^+$ (2).

Theorem 1: If f has an n th Peano derivative, $f_n(x)$, at each x in $\mathbb{R}$ with infinite values allowed, then f_n is a function of Baire class one.

(This theorem originally appeared in [1] but with an invalid proof.)

To establish further properties of such functions f_n the following auxiliary theorem is useful and of interest in its own right.

Theorem 2: If $f_n(x)$ exists for all x in $\mathbb{R}$ with infinite values allowed, and if f_n is bounded above or below on an interval I, then $f_n = f^{(n)}$, the ordinary n th derivative of f, on I.

This result can be established by copying the proof of the corresponding assertion for the finite case from [2], [4] or [5] and making the necessary minor changes. We chose the last of these three since it required only a small modification in a lemma.

Using Theorem 2 we establish the following properties of Peano derivatives.