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On a Problem Concerning L^p Moduli of Smoothness

THEOREM. *Let* (a,b) *be a finite or infinite interval,* $\beta \geq 1$ *an integer,* $\alpha > 0$ *and* $1 \leq p < \infty$, $f \in L^p(a,b)$. *If*

(1) $\qquad \|\Delta_{t_n}^{\beta} f\|_{L^p(a,b-\beta t_n)} \leq t_n^{\alpha}$

for a sequence $\{t_n\} \to 0+0$ *satisfying*

(2) $\qquad \{t_n/t_{n+1}\} = O(1) \qquad (n \to \infty)$, *then*

(3) $\qquad \|\Delta_h^{\beta} f\|_{L^p(a,b-\beta h)} = O(h^{\alpha}) \qquad (h \to 0+0)$.

Here,

$$\Delta_h^{\beta} f(x) = \sum_{n=1}^{\beta} (-1)^{\beta+i} \binom{\beta}{i} f(x+ih).$$

The analogous result in $C_{2\pi}$ was proved by De Vore for $\beta=2$, and he raised the question if the same is true in the L^p norm. Freud verified this in the case $p=2$ and Ditzian for every $p \geq 1$. Freud also showed that (2) is necessary for the implication (1)$\Rightarrow$(3) when $\alpha < \beta$ and the problem of the sufficiency of (2) for every $\beta \geq 1$ was posed by Ditzian. Boman solved this problem in a very general setting. Since Boman's approach is rather abstract and heavily uses the translation invariance of $L^p(-\infty,\infty)$, it seems worthwhile to give a direct proof which also applies to the finite interval.