Roy O. Davies, Department of Mathematics, The University,
Leicester, LE1 7RH, England

Symmetric Sets are Measurable

Evans and Weil [1] call a set $E \subseteq \mathbb{R}$ <u>symmetric</u> if its characteristic function f satisfies the condition: for every $x \in \mathbb{R}$ there exists δ_x such that $f(x + h) = f(x - h)$ for $0 < h < \delta_x$; and they ask whether if E is symmetric it must be measurable. The answer is Yes, and indeed either E or its complement must contain a co-countable open set.

Hausdorff [4] called a real function f <u>symmetrically continuous</u> if for every $x \in \mathbb{R}$, $f(x + h) - f(x - h) \to 0$ as $h \to 0$, and he asked whether a symmetrically continuous function can have uncountably many discontinuity points. In this connection Fried [3] proved that it has a dense set of continuity points. Now the characteristic function f of a symmetric set is evidently a symmetrically continuous function, so by Fried's theorem it has a continuity point. For a characteristic function, this means that there is a non-degenerate open interval on which f is constant. Let c be an interior point of this interval, and let d be the supremum of points $x \in \mathbb{R}$ such that f is constant on a co-countable open subset of (c, x) . We see that $d = +\infty$ because otherwise the symmetry property applied at d leads to a contradiction. We argue similarly on the left of c , and conclude that f is constant on a co-countable open subset of $\mathbb{R}$.

Readers may be interested in what Fried's argument reduces to in the case of our characteristic function f . Let B_n