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The Uniform Continuity of Certain Translation Semigroups

Let $L^2(R^+;K)$ be the Lebesgue space of square summable functions f on the positive reals with values in a separable Hilbert space K. That is, f satisfies

i. $\langle f(x),k\rangle$ is measurable, a.e.(x), for each $k \in K$,

ii. $\int_0^\infty ||f(x)||^2 dx < \infty$, where $||f(x)||$ denotes the norm of f(x) in K.

The inner product is given by $\int_0^\infty \langle f(x),g(x)\rangle dx$, where $\langle f(x),g(x)\rangle$ is the inner product in K. For each $h \geq 0$, we define the translation operator S_h by

$$S_h f(x) = f(x + h).$$

$\{S_h\}$ is a strongly continuous semigroup of operators, i. e. for each f, $||S_h f - f||$ converges to 0 as $h \to 0^+$. However, it fails to be uniformly continuous; that is, $||S_h - I||$ does <u>not</u> converge to zero as $h \to 0^+$. For example,

$$f_h(x) = \begin{cases} 1/\sqrt{h} & \text{for } 0 \leq x \leq h \neq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

satisfies $||f_h|| = 1$, $S_h f_h = 0$, so that $||S_h f_h - f_h|| = 1$. Hence $||S_h - I|| \geq 1$ for all $h > 0$. But if S_h is restricted to multiples of $e_1(x) = e^{-x}$, then since $S_h e_1 = e^{-h} e_1$, we have $||S_h - I|| = |e^{-h} - 1| \to 0$ as $h \to 0^+$. Therefore, one may ask the following question: If L is a closed linear subspace of $L^2(R^+;K)$ satisfying $S_h(L) \subset L$