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Summability of Approximate Derivatives

Let $F: [0,1] \to R$ be approximately differentiable with finite approximate derivative F'_{ap}. If $DF = \{x | F'(x) \text{ exists}\}$ and ΔF denotes the interior of DF, then ΔF is a dense open set [2]. Moreover, if F is not everywhere differentiable in the ordinary sense and M is any positive integer, there is a component of ΔF on which F' takes on both M and -M [3]. Thus if F'_{ap} is "well-behaved" on ΔF, one might expect it to be "well-behaved" on [0,1]. For example, if F'_{ap} is bounded above or below on ΔF, then DF=[0,1]. In [1] it is shown that the summability of F'_{ap} over ΔF does not imply its summability over [0,1]. In the positive direction, it is shown that the natural "test set" for summability is $\Delta^* F$, which is the union of all open intervals (a,b) such that F is continuous at each point of (a,b) and F'(x) exists almost everywhere on (a,b). The results and examples are as follows.

Example 1. There exists an unbounded, approximately differentiable function F, such that $F'_{ap}(x)$ is summable over ΔF.

Example 2. There exists on approximately differentiable ACG_* function F, such that $F'_{ap}(x)$ is summable over ΔF but not over [0,1]. Moreover, there is an everywhere differentiable function G such that F(x) = G(x) on $[0,1] \setminus \Delta F$.