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Generalizations of L'Hospital's Rule

The following is a version of the well-known l'Hôpital rule.

Theorem 1. Let F, G be two real-valued functions defined on the open interval (a,b). Suppose that the following conditions are satisfied:

(H1) F, G are differentiable on (a,b) and $G'(x) \neq 0$ for all x in (a,b);

(H2) $\lim_{x\to a+} [F'(x)/G'(x)] = A$;

(H3) $\lim_{x\to a+} F(x) = 0 = \lim_{x\to a+} G(x)$.

Then

(C) $\lim_{x\to a+} [F(x)/G(x)] = A$.

It is known (cf. [4]) that the existence of the ordinary limit in (H2) cannot be weakened to that of the approximate limit even if one only wants to conclude that $\text{aplim}_{x\to a+} [F(x)/G(x)] = A$. Sandwiched in between the concept of the ordinary limit and that of the approximate limit, a concept called the essential limit is introduced, and using it we show that not only the condition (H2) but also the conditions (H1) and (H3) can be weakened so that the conclusion (C) still holds true. In fact, we have the following results:

Theorem I. Let a,b be two extended real numbers with $-\infty \leq a < b \leq +\infty$, and let F,G be real-valued functions defined on the open interval (a,b). Suppose that the following conditions are satisfied:

(h1) the approximate derivatives $F_{(1)}(x)$ and $G_{(1)}(x)$ exist finitely and $G_{(1)}(x) > 0$ for almost all x in (a,b);

(h2) $\text{esslim}_{x\to a+} [F_{(1)}(x)/G_{(1)}(x)] = A$;