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The Space of Bounded Derivatives

Let R denote the real line and let

$$D = \{f:R \to R: \ f \text{ is bounded and there is a function } F \text{ such that } f(x) = F'(x) \text{ for all } x \in R\}.$$

For each $f,g \in D$ set

$$d(f,g) = \sup_{x \in R} |f(x) - g(x)|.$$

Then d is a metric on D and convergence in d is uniform convergence. So a standard theorem of advanced calculus says that D is complete. Furthermore, D is a vector space if addition of functions and multiplication of functions by real numbers are defined in the usual way.

As a follow-up to papers on nowhere monotone, differentiabl functions by Goffman and by Katznelson and Stromberg, existence of such functions was established by applying the Baire Category Theorem to an appropriate subspace of D (see [3]). The results here also expand on a paper by Goffman [2] in which he gives a short construction of a bounded derivative which is not Riemann integrable. Another construction can be found on page 26 of the excellent expository article by Bruckner and Leonard [1]. From Theorem 1 it follows, using the Baire Category Theorem, that there are bounded derivatives that are not Riemann integrable on any subinterval of R. Theorem 2 proves even more; namely, that there are bounded derivatives which are discontinuous almost everywhere.