Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 276–280

Steven N. Evans and J.W. Pitman Department of Statistics, University of California at Berkeley, 367 Evans Hall Berkeley, CA 94720, U.S.A.*

Does Every Borel Function Have a Somewhere Continuous Modification?

Is it possible to place any limits on the roughness of a Borel measurable function from the set of real numbers into itself? One answer to this question is provided by Lusin's theorem (see, for example, §8.29 of [9]): if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Borel, then for every $\epsilon > 0$ there is a continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\lambda(\{x : f(x) \neq g(x)\}) < \epsilon$, where λ is Lebesgue measure. In some sense, Lusin's theorem is the best possible result of this type. It is certainly not the case that any Borel function is equal almost everywhere to a continuous function, as shown by the trivial example

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What happens, however, if we look in a different direction and ask whether an arbitrary Borel function is almost everywhere equal to a function that is continuous at at least one point? That the answer to this question is negative is implied by a striking example of Carathéodory ([5], §§427 - 428) that exhibits a Borel function with the following remarkable property.

Definition. *A Borel function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the property (Car) if for every non-empty, open set $A \subset \mathbb{R}$ and every non-empty, open set $B \subset \mathbb{R}$, the set $f^{-1}(A) \cap B$ has non-zero Lebesgue measure.*

It is clear that if f is any function with property (Car) and x is an arbitrary point in $\mathbb{R}$, then there is no function g such that $f = g$ almost everywhere and g is continuous at x.

Carathéodory's construction is quite involved and uses some relatively deep facts about singularities of functions of a complex variable. Berman [2] offers another, somewhat simpler, construction of a function with the property (Car). Indeed, Berman shows that his example displays the following, even more erratic, behaviour.

*The first author is a Presidential Young Investigator and the second was supported in part by NSF grant MCS91-07531.

Received by the editors May 20, 1992