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Products of Darboux Functions

Let us establish some terminology to be used. R denotes the set of all reals. I denotes a non-degenerate closed interval. If A is a planar set, we denote its x-projection by dom(A) and y-projection by rng(A).

We shall consider real functions defined on a real interval. No distinction is made between a function and its graph. The symbols $C^-(f,x)$ and $C^+(f,x)$ denote the left and right cluster sets of f at the point x. The symbol $C(f)$ denotes the set of all continuity points of f. The notation $[f > 0]$ means the set $\{x : f(x) > 0\}$. Likewise for $[f = 0]$, $[f \neq 0]$, etc. For subsets $A, B \subset R$ let $\mathcal{D}^*(A, B)$ denote the class of all functions $f : A \longrightarrow B$ such that $\text{cl}_A f^{-1}(y) = A$ for each $y \in B$. Let us remark that if A is an F_σ set and A is c dense-in-itself, then the class $\mathcal{D}^*(A, R)$ contains Baire 2 functions (see [2]).

The function f is said to be Darboux if $f(C)$ is connected whenever C is a connected subset of the domain of f. If each open set containing f also contains a continuous function with the same domain as f, then f is almost continuous [2]. It is clear that if $f : I \longrightarrow R$ is almost continuous, then f is connected and, therefore, it has the Darboux property. Moreover, if f meets each closed subset F of $I \times R$ with $\text{int}(\text{dom}(F)) \neq \emptyset$, then f is almost continuous [2].

We shall use the following set-theoretical assumption.

$A(c)$ – the union of less than 2^ω many first category subsets of R is of the first category again.

Note that this statement is a consequence of Martin's Axiom and therefore also the Continuum Hypothesis (see e.g. [2]).

It is well known that each real-valued function defined on a real interval can be expressed as a sum of two Darboux functions [2]. This fact was improved by Fast in the following way: if $\mathcal{F}$ is a collection of c-many real functions then there exists a function g such that $f + g$ is Darboux for each $f \in \mathcal{F}$ [2]. In 1967, Mišik proved that for each countable family $\mathcal{F}$ of Baire α functions (where $\alpha > 1$) there exists a Baire α function g such that $f + g$ has the Darboux property for every $f \in \mathcal{F}$ [2]. In 1984, Pu and Pu proved the analogous result for finite

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