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Metric Space of Metrics Defined on a Given Set

1. Introduction

Let X be a non-void set. Denote by $\mathcal{M} = \mathcal{M}(X)$ the set of all metrics on X. We can introduce a metric d^* on $\mathcal{M}$ as follows:

If $d, d' \in \mathcal{M}$ then

$$d^*(d, d') = \min \{1, \sup_{x,y \in X} |d(x,y) - d'(x,y)|\}.$$

First of all recall some basic definitions and notations.

The symbol $t_\alpha (\alpha > 0)$ stands for a trivial metric on X i.e. $t_\alpha(x,x) = 0$ for every $x \in X$ and $t_\alpha(x,y) = \alpha$ for $x \neq y, x, y \in X$. Furthermore if $d \in \mathcal{M}$ and $\varepsilon > 0$, denote by $K(d,\varepsilon) = \{d' \in \mathcal{M} : d^*(d,d') < \varepsilon\}$ (a ball in $\mathcal{M}$) and $\overline{K}(d,\varepsilon) = \{d' \in \mathcal{M} : d^*(d,d') \leq \varepsilon\}$ (a closed ball in $\mathcal{M}$).

Denote by $|B|$ the cardinality of the set B and by $\mathcal{P}(B)$ the power set of B.

If $|X| = 1$, then obviously $|\mathcal{M}(X)| = 1$. Therefore in the following we shall always assume that $|X| \geq 2$.

Denote by $\aleph_0$ and c the cardinality of the set of all positive integers $\mathbb{N}$ and the set of all real numbers $\mathbb{R}$, respectively.

If $\mathcal{M}_1 \subset \mathcal{M}$, then $\mathcal{M}_1$ is considered as a metric space with the metric $d^*|_{\mathcal{M}_1 \times \mathcal{M}_1}$ (a metric subspace of $\mathcal{M}$).

2. Lemmas

In our further considerations some lemmas will take an important place. It is easy to check, that the space $(\mathcal{M}, d^*)$ is not complete. For example, the sequence $\{t_{\frac{1}{k}}\}_{k=1}^{\infty}$ of elements of $\mathcal{M}$ is fundamental, nevertheless has no limit in $\mathcal{M}$. In connection with this we mention some subspaces of $\mathcal{M}$, which are already complete. Suppose $\alpha > 0$ and put