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Algebraic Properties of $\mathcal{E}$–continuous Functions

1. Introduction.

Let $x \in \mathbb{R}$. A *path leading to* x is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and x is a point of bilateral accumulation of E_x. For $x \in \mathbb{R}$ let $\mathcal{E}(x)$ be a family of paths leading to x. A *system of paths* is a collection $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ such that each $E_x \in \mathcal{E}(x)$ for every $x \in \mathbb{R}$ (compare with [1]). Sometimes we shall simply refer to E_x as a "path."

We say that L_x (R_x) is a *left (right) path leading to* x if $L_x = E_x \cap (-\infty, x]$ $(R_x = E_x \cap x, \infty))$ for some path $E_x \in \mathcal{E}(x)$.

We shall only consider system of paths $\mathcal{E}$ having the property that if L_x is a left path leading to x and R_x is a right path leading to x then $L_x \cup R_x$ is an element of $\mathcal{E}(x)$, and we shall assume that $\mathbb{R} \in \mathcal{E}(x)$ for each $x \in \mathbb{R}$. We shall classify systems of paths according to the following scheme: a system of paths $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ will be said to be

– of *δ-type*, if $E_x \cap [x-\delta, x+\delta]$ contains a path in $\mathcal{E}(x)$ for every $E_x \in \mathcal{E}(x)$ and for every $\delta > 0$.

– of *σ-type*, if $\mathcal{E}$ is a δ-type system of paths, and for each triple of sequences of numbers $(a_n)_{n=1}^{\infty}$, $(x_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $b_{n+1} < a_n < x_n < b_n$, $(a_n < x_n < b_n < a_{n+1})$ $b_n \searrow x$ $(a_n \nearrow x)$ and for each left or right or bilateral paths $E_{x_n} \subset [a_n, b_n]$ leading to x_n for $n \in \mathbb{N}$, the set $\bigcup_{n=1}^{\infty} E_{x_n} \cup \{x\}$ contains a right path R_x (left path L_x) derived from an $E_x \in \mathcal{E}(x)$.

– of *c-type*, if $\mathcal{E}$ is a σ-system of paths and every Cantor set C_x such that x is a bilateral point of accumulation of C_x, belongs to $\mathcal{E}(x)$.

Such systems will be called shortly *δ-systems*, *σ-systems* and *c-systems*, respectively. We consider real functions of a real variable, unless otherwise explicitly stated.

Let X be a topological space, let $f : \mathbb{R} \to X$, and let $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ be a system of paths. We say that *function f is $\mathcal{E}$–continuous at x (f has a path at x)* if there exists a path $E_x \in \mathcal{E}(x)$ such that $f : E_x$ is continuous at x. If f is $\mathcal{E}$–continuous at every point x, then we say that f is *$\mathcal{E}$–continuous*.

We say that function f *has a left (right) path at* x if there exists a left (right)