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Closure of Darboux Graphs

What is the nicest class of functions with the property that the graph of any Darboux function would have the same closure as some member of this class? In 1974, Hugh Miller [6] showed that the graph of any Darboux function $f : I \to I$, where $I = [0,1]$, has the same closure in I^2 as the graph of some connectivity function $g : I \to I$. Using an analogous transfinite induction argument, he improved this result to obtain that $\bar{f} = \bar{h}$ for some almost continuous function $h : I \to I$ (unpublished). In 1990, at the Seventh Annual Auburn Miniconference on Real Analysis, Ken Kellum asked whether the above results can be generalized so that the function g in Miller's theorem can be chosen to be a connectivity function extendable to a connectivity function from I^2 into I. In this note, we use another technique like in [4] and [3] to show the answer is yes. To illustrate that Miller's result does not generalize to I^2, Kellum gave an example of a Darboux function $f : I^2 \to I^2$ for which $\bar{f} = \bar{h}$ for no almost continuous function $h : I^2 \to I^2$. We end with an equivalence between the uniform closure of the class of Darboux functions and the closure of Darboux graphs.

Let $f : X \to Y$. Then f is *Darboux* (*connectivity*) if $f(C)$ (the graph of $f|C$) is connected for every connected subset C of X. We say f is *peripherally continuous* at x if for each open neighborhood U of x and V of $f(x)$, there is an open neighborhood W of x in U such that $f(\mathrm{bd}(W))\imath V$. We say f is *almost continuous* if each open neighborhood of the graph of f in $X \times Y$ contains the graph of a continuous function $g : X \to Y$. A connectivity function $f : I \to I$ is said to be *extendable* if there is a connectivity function $g : I^2 \to I$ such that for all $x \in I$, $g(x,0) = f(x)$. For functions from I into I, we have:

$$\text{extendable} \implies \text{almost continuous} \implies \text{connectivity} \implies \text{Darboux}$$

where the first arrow is from [8, Cor. 1, Prop. 2] and the second is from [8, Cor., p. 261]. But for functions from I^n into I^m, $n \geq 2$, we have:

$$\text{peripherally continuous} \iff \text{connectivity} \implies \text{almost continuous}$$

where $\iff$ is from [5, Th. 1] or [9, Cor.] and [8, Th. 4] and $\implies$ is from [8, Cor. 1].

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