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On the Relative Grid Dimension of Continuous Functions

The n^{th} grid G_N on the unit square $S = [0,1] \times [0,1]$ is the se t of elementary closed squares of the regular $n \times n$ subdivision of S. For any $E \subset S$, $E \neq \phi$ let $N(E,n)$ denote the number of elements of G_n which meet E. For a given subsequence of natural numbers $\nu(n)$ $(n = 1, 2, \ldots)$ the grid dimension $\alpha_\nu(E)$ of a set E relative to the sequence ν is defined by

$$\begin{aligned}\alpha_\nu(E) &= \inf\{\alpha : \limsup_{n\to\infty} \frac{N(E,\nu(n))}{\nu(n)^\alpha} < \infty\} \\ &= \sup\{\alpha : \limsup_{n\to\infty} \frac{N(E,\nu(n))}{\nu(n)^\alpha} = \infty\},\end{aligned}$$

or equivalently

$$\alpha_\nu(E) = \limsup_{n\to\infty} \frac{\log N(E,\nu(n))}{\log \nu(n)}. \qquad (*)$$

For $\nu(n) = n$ $(n = 1, \ldots)$ we put $\alpha_\nu(E) = \alpha(E)$ and this number is called the grid dimension. Obviously, $0 \leq \alpha_\nu(E) \leq \alpha(E) \leq 2$ for any ν and $E \neq \phi$. In this paper we study the growth conditions on ν implying $\alpha_\nu(E) = \alpha(E)$ for any E at one hand, and on the other, with special attention to the case when $E = \Gamma_f$, the graph of a continuous function f.

The exact value of the rarefaction index

$$\tau = \inf\{t : \text{ the grid dimension was known in the year } t\}$$

is not known, but certainly $\tau \leq 1928$. This dimension, perhaps the first time was used by Bouligand in [BO], 1928 (see also [MA] for references). As it turns out, it has been reintroduced by several authors, each giving to it a new name (see [FA], p. 38), and as a result, this single concept now enjoys such a long list of titles that seeing it, even a Spanish Grandee could turn green with envy. It is

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