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Hans P. Heinig, Department of Mathematics and Statistics, McMaster University, Hamilton, L8S 4K1, Canada

Weighted Inequalities in Function Spaces

The problem of characterizing non-negative locally integrable (weight) functions u and v on $\mathbb{R}^+$, for which Hardy's inequality

$$\left\{\int_0^\infty u(x)\left[\int_0^x f(t)dt\right]^q dx\right\}^{\frac{1}{q}} \leq C\left\{\int_0^\infty v(x)f(x)^p dx\right\}^{\frac{1}{p}}, 0 < p, q < \infty, \quad (1)$$

holds for all non-negative $f \in L_v^p$ has been completely solved. The formulation for the discrete version of this result in the index range $1 < p, q < \infty$ is

Theorem 1 ([1],[4]) *Suppose $1 < p, q < \infty$ and $\{u_k\}_{k\in\mathbb{N}}$, $\{v_k\}_{k\in\mathbb{N}}$ are sequences such that $u_k \geq 0$, $v_k > 0$, $k \in \mathbb{N}$. Then there exists a constant $B > 0$, such that*

$$\left\{\sum_{n\in\mathbb{N}} u_n \left[\sum_{k=1}^n a_k\right]^q\right\}^{\frac{1}{q}} \leq B\left\{\sum_{n\in\mathbb{N}} v_n a_n^p\right\}^{\frac{1}{p}} \quad (2)$$

holds for all non-negative sequences $\{a_k\} \in \ell^p_{\{v_n\}}$, if and only if

(i) in case $1 < p \leq q < \infty$

$$B_1 = \sup_{m\in\mathbb{N}}\left\{\sum_{n=m}^\infty u_n\right\}^{\frac{1}{q}}\left\{\sum_{n=1}^m v_n^{1-p'}\right\}^{\frac{1}{p'}} < \infty$$

(ii) in case $1 < q < p < \infty$

$$B_2 = \left\{\sum_{m\in\mathbb{N}}\left[\sum_{n=m}^\infty u_n\right]^{\frac{r}{q}}\left[\sum_{n=1}^m v_n^{1-p'}\right]^{\frac{r}{q'}} v_m^{1-p'}\right\}^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and p', q' are the conjugate indices of p and q, respectively.

In this talk we discuss weight characterizations of inequalities of the form (1) and (2), where the weighted Lebesgue spaces are replaced by more general