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Generalized Lebesgue Points *

Let f be Lebesgue integrable on $[-\pi, \pi]$ and let $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + \sin nt)$, be the Fourier series of f. A well known result of Lebesgue, states that if x is a Lebesgue point of f, then

$$\lim_{n\to\infty} \sigma_n(x) = f(x), \qquad (*)$$

where we adopt the usual notation $\sigma_n(x) = \frac{s_0(x)+s_1(x)\cdots+s_{n-1}(x)}{n}$, and $s_0(x) = \frac{a_0}{2}$, $s_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt)$, $m = 1, 2 \ldots$. In the paper being summarized here we prove that if f is Denjoy-Perron integrable on $[-\pi, \pi]$, and if x is a generalized Lebesgue point of f, then (*) holds. Since it is shown that a Lebesgue point of a Lebesgue integrable function is a generalized Lebesgue point this result gives a generalization of the classical result. Prior to giving a precise statement of the main result, we list some background results.

A well-known result of Féjer states that if f is continuous at x, then its Fourier series is $(C, 1)$-summable at x to $f(x)$. Over the years this continuity condition has been relaxed. Lebesgue showed that if f is Lebesgue integrable, then its Fourier series is $(C, 1)$-summable to $f(x)$ at any point where $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(x) - f(t)| dt = 0$, i.e. at the full measure set of **Lebesgue points** of f. Fatou showed that if all we are concerned with is A-summability, then $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h}(f(x) - f(t)) dt = 0$, will do. These are called points of **C-continuity**. What happens if the function is only D^*-integrable? There is a theorem of Marcinkiewicz quoted in Čelidze & Džvaršeĭšvili as: If a function is D-integrable, then it is $(C, 1)$-summable at all points of C-continuity. This seems to be a misquote since then we could extend Fatou's result to $(C, 1)$-summability. There is an example in Zygmund of a function whose differentiated series is not $(C, 1)$-summable at a point where it has a derivative; however it is not clear that this function is ACG^*, or even ACG. A reading of the original paper of Marcinkiewicz seems to suggest the correct result is: A D-integrable function is $(C, 1)$-summable at almost all points of C-continuity. In the case of Lebesgue integrable functions this implies a weaker form of Lebesgue's result.

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