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SPLITTING AND NONSPLITTING, II: A LOW$_2$ C.E. DEGREE ABOVE WHICH $\boldsymbol{0'}$ IS NOT SPLITTABLE

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Abstract. It is shown that there exists a low$_2$ Harrington non-splitting base — that is, a low$_2$ computably enumerable (c.e.) degree $\boldsymbol{a}$ such that for any c.e. degrees $\boldsymbol{x}, \boldsymbol{y}$, if $\boldsymbol{0'} = \boldsymbol{x} \vee \boldsymbol{y}$, then either $\boldsymbol{0'} = \boldsymbol{x} \vee \boldsymbol{a}$ or $\boldsymbol{0'} = \boldsymbol{y} \vee \boldsymbol{a}$. Contrary to prior expectations, the standard Harrington non-splitting construction is incompatible with the low$_2$-ness requirements to be satisfied, and the proof given involves new techniques with potentially wider application.

§1. **Introduction.** We say that a set $A \subseteq \omega$ is *computably enumerable* (c.e.) if there is an algorithm to enumerate the elements of it. For $A, B \subseteq \omega$, we say that A is *Turing reducible to* (or *computable* in) B if there is an algorithm to decide for every $x \in \omega$, whether or not $x \in A$ when given answers to all questions of the form " Is $y \in B$?". We use $A \leq_{\mathrm{T}} B$ to denote that A is Turing reducible to B, and we write $A \equiv_{\mathrm{T}} B$ if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. A (Turing) *degree* is an equivalence class of A under $\equiv_{\mathrm{T}}$ for some $A \subseteq \omega$. We say that a degree $\boldsymbol{a}$ is *computably enumerable* (c.e.) if it contains a c.e. set. Post [10] pioneered the study of the structure of the c.e. degrees. He observed that there is a greatest c.e. degree $\boldsymbol{0'}$, and asked whether or not there is a c.e. degree other than $\boldsymbol{0}$ (the least degree) and $\boldsymbol{0'}$.

Friedberg [5], and independently Muchnik [8] answered Post's question affirmatively. Furthermore, Sacks [12, 13], showed that:

THEOREM 1.1 (Sacks Splitting Theorem [12]). *For any c.e. degree $\boldsymbol{a} \neq \boldsymbol{0}$, there exist c.e. degrees $\boldsymbol{a_0}, \boldsymbol{a_1}$ such that $\boldsymbol{a_0}, \boldsymbol{a_1} < \boldsymbol{a}$ and $\boldsymbol{a} = \boldsymbol{a_0} \vee \boldsymbol{a_1}$.*

THEOREM 1.2 (Sacks Density Theorem [13]). *For any c.e. degrees $\boldsymbol{b} < \boldsymbol{a}$, there is a c.e. degree $\boldsymbol{c}$ such that $\boldsymbol{b} < \boldsymbol{c} < \boldsymbol{a}$.*

A basic question was then whether or not Theorems 1.1 and 1.2 could be combined, which was eventually answered negatively by Lachlan [7].

THEOREM 1.3 (Lachlan Nonsplitting Theorem [7]). *There exist c.e. degrees $\boldsymbol{b} < \boldsymbol{a}$ such that for any c.e. degrees $\boldsymbol{x}, \boldsymbol{y}$, if $\boldsymbol{x} \vee \boldsymbol{y} = \boldsymbol{a}$, then either $\boldsymbol{a} \leq \boldsymbol{x} \vee \boldsymbol{b}$ or $\boldsymbol{a} \leq \boldsymbol{y} \vee \boldsymbol{b}$.*

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