THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 4, Dec. 2002

CARDINALITIES IN THE PROJECTIVE HIERARCHY

GREG HJORTH

CONTENTS

1. Introduction 1351
2. Outline 1352
3. Notational issues 1353
4. Hairy $\utilde{\mathbf{\Sigma}}^1_1$ set forcing: $\mathscr{P}_{\mathscr{H}}$ 1354
5. And its amalgam: $\mathscr{P}_{\mathscr{H}} \otimes_{S_1,S_2} \mathscr{P}_{\mathscr{H}}$ 1355
6. Some dense sets for $\mathscr{P}_{\mathscr{H}} \otimes_{S_1,S_2} \mathscr{P}_{\mathscr{H}}$ 1358
7. Some things to do with a generic 1365
8. Generalities on finding generics 1368
9. Finishing up 1370

§1. Introduction. We show that the "effective cardinality" of the collection of $\utilde{\mathbf{\Pi}}^1_{n+1}$ sets is strictly bigger than the effective cardinality of the $\utilde{\mathbf{\Pi}}^1_n$. The phrase *effective cardinality* is vague but can be made exact in the usual ways. For instance:

THEOREM 1.1. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then in* $L(\mathbb{R})$ *there is no injection*

$$i : \utilde{\mathbf{\Pi}}^1_{n+1} \hookrightarrow \utilde{\mathbf{\Pi}}^1_n.$$

A few years ago Tony Martin showed a similar result, establishing the non-existence of an injection from $\utilde{\mathbf{\Pi}}^1_m$ to $\utilde{\mathbf{\Pi}}^1_n$ for m sufficiently larger than n. His method did not seem to work for $m = n + 1$.

This present paper gives level by level calculations for the projective hierarchy, but it too falls short of a complete analysis, in as much as it leaves the position of the effective cardinals in the Wadge degrees largely obscure. At the low levels it takes some time for any new cardinals to appear. Whenever Γ_1, Γ_2 are non-trivial Wadge degrees strictly included in $\utilde{\mathbf{\Delta}}^0_2$ one has

$$|\Gamma_1|_{L(\mathbb{R})} = |\Gamma_2|_{L(\mathbb{R})}.$$

Beyond $\utilde{\mathbf{\Delta}}^0_2$ it is known from [3] that the different levels of the Borel hierarchy have distinct effective cardinalities. It is unclear whether there might be cardinals lying strictly between say $\utilde{\mathbf{\Pi}}^0_2$ and $\utilde{\mathbf{\Pi}}^0_3$, though Itay Neeman has proved that there is no $L(\mathbb{R})$ injection from $\utilde{\mathbf{\Pi}}^0_{\alpha+2}$ to $\utilde{\mathbf{\Delta}}^0_{\alpha+2}$. Thus as a landmark of our ignorance:

QUESTION 1. Is $|\utilde{\mathbf{\Pi}}^0_2|_{L(\mathbb{R})} < |\utilde{\mathbf{\Delta}}^0_3|_{L(\mathbb{R})}$?

Received January 5, 2001; revised January 28, 2002.

0022-4812/02/6704-0007/$3.20