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$0^{\#}$ AND INNER MODELS

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§1. In this paper we examine the cardinal structure of inner models that satisfy GCH but do not contain $0^{\#}$. We show, assuming that $0^{\#}$ exists, that such models necessarily contain Mahlo cardinals of high order, but without further assumptions need not contain a cardinal κ which is κ-Mahlo. The principal tools are the Covering Theorem for L and the technique of reverse Easton iteration.

Let I denote the class of Silver indiscernibles for L and $\langle i_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ its increasing enumeration. Also fix an inner model M of GCH not containing $0^{\#}$ and let ω_α denote the ω_α of the model $M[0^{\#}]$, the least inner model containing M as a submodel and $0^{\#}$ as an element.

THEOREM 1.1. *Suppose that α is greater than* 0. (*a*) *$i_{\omega_1\cdot\alpha}$ is an M-cardinal, and unless α is a limit ordinal of countable $M[0^{\#}]$-cofinality, so is its L-cardinal successor.*

(*b*) *If β is less than $i_{\omega_1^{L[0^{\#}]}\cdot\omega}$ then there is a proper inner model M of $L[0^{\#}]$ satisfying GCH in which the only ordinals between ω and β which are M-cardinals are those which are required to be by part* (*a*).

It follows from (a) that for finite n, ω^M_{2n+1} is at most $i_{\omega_1\cdot(n+1)}$ and that ω^M_{2n+2} is at most the L-cardinal successor to $i_{\omega_1\cdot(n+1)}$. It follows from (b) that these bounds are optimal. The restriction in (b) on β cannot be weakened, as otherwise an increasing ω-sequence of Silver indiscernibles, and hence $0^{\#}$ itself, would belong to M. In fact the supremum of the $i_{\omega_1\cdot n}$'s must be large in M:

THEOREM 1.2. (*a*) *$i_{\omega_1\cdot\alpha}$ is inaccessible in M for limit α.*

(*b*) *If β is less than $i_{\omega_1^{L[0^{\#}]}\cdot\omega\cdot\omega}$ then there is a proper inner model M of $L[0^{\#}]$ satisfying GCH in which the only ordinals less than β which are M-inaccessible are those which are required to be by part* (*a*).

It follows from (a) that for finite n, the n-th M-inaccessible is at most $i_{\omega_1\cdot\omega\cdot n}$. It follows from (b) that these bounds are optimal. As before, the restriction in (b) on β cannot be weakened, as otherwise $0^{\#}$ would belong to M.

We can also obtain Mahlo cardinals of high order in M. Define: κ is 0-*Mahlo* (or simply *Mahlo*) iff the set of inaccessible $\bar\kappa < \kappa$ is stationary in κ, κ is $\alpha + 1$-*Mahlo* iff the set of α-Mahlo $\bar\kappa < \kappa$ is stationary in κ, and for limit λ, κ is λ-*Mahlo* iff κ is α-Mahlo for every $\alpha < \lambda$.

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