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MORE UNDECIDABLE LATTICES OF STEINITZ EXCHANGE SYSTEMS

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Abstract. We show that the first order theory of the lattice $\mathscr{L}^{<\omega}(S)$ of finite dimensional closed subsets of any nontrivial infinite dimensional Steinitz Exhange System S has logical complexity at least that of first order number theory and that the first order theory of the lattice $\mathscr{L}(S_\infty)$ of computably enumerable closed subsets of any nontrivial infinite dimensional computable Steinitz Exchange System S_∞ has logical complexity exactly that of first order number theory. Thus, for example, the lattice of finite dimensional subspaces of a standard copy of $\oplus_\omega \boldsymbol{Q}$ interprets first order arithmetic and is therefore as complicated as possible. In particular, our results show that the first order theories of the lattice $\mathscr{L}(V_\infty)$ of c.e. subspaces of a fully effective $\aleph_0$-dimensional vector space V_∞ and the lattice of c.e. algebraically closed subfields of a fully effective algebraically closed field F_∞ of countably infinite transcendence degree each have logical complexity that of first order number theory.

§1. Introduction and notation. In the late 1970's Chris Ash asked if the first order theory of the lattice of algebraically closed subfields of an algebraically closed field K of infinite transcendence degree is decidable. This question was answered strongly in the negative by Magidor, Rosenthal, Rubin, and Srour [21]. Not only does that paper show that the first order theory of the above lattice has the complexity of full second order logic on the transcendence degree of K, but that paper also shows far more generally that the first order theory of the lattice of closed subsets of any nontrivial infinite dimensional nontrivial Steinitz Exchange System is at least as complex as second order number theory. These results lead to questions about the decidability of lattices which arise from various restricted classes of closed subsets. Of particular interest are the lattice of finite dimensional closed subsets of a nontrivial Steinitz Exchange System and the lattices obtained when some effective structure (i.e., computability conditions) is imposed on the Steinitz Exchange System.

In the current paper we first show that the first order theory of the lattice of closed *finite* dimensional subsets of *any* nontrivial infinite dimensional Steinitz Exchange System is undecidable by encoding first order number theory into the first order theory of the lattice. Our encoding actually provides us with an m-reduction Namely, we prove:

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Dedicated to the memory of Chris Ash