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ON MODAL LOGICS BETWEEN $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ AND $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$

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Abstract. We prove that every n-modal logic between $\mathbf{K}^n$ and $\mathbf{S5}^n$ is undecidable, whenever $n \geq 3$. We also show that each of these logics is non-finitely axiomatizable, lacks the product finite model property, and there is no algorithm deciding whether a finite frame validates the logic. These results answer several questions of Gabbay and Shehtman. The proofs combine the modal logic technique of Yankov–Fine frame formulas with algebraic logic results of Halmos, Johnson and Monk, and give a reduction of the (undecidable) representation problem of finite relation algebras.

§1. Introduction and results. Here we deal with axiomatization and decision problems of *n-modal logics*: propositional multi-modal logics having finitely many unary modal operators $\Diamond_0, \dots, \Diamond_{n-1}$ (and their duals $\Box_0, \dots, \Box_{n-1}$), where n is a non-zero natural number. Formulas of this language, using propositional variables from some fixed countably infinite set, are called *n-modal formulas*. Frames for n-modal logics—*n-frames*—are structures of the form $\mathscr{F} = (F, R_0, \dots, R_{n-1})$ where R_i is a binary relation on F, for each $i < n$. A *model* on an n-frame $\mathscr{F} = (F, R_0, \dots, R_{n-1})$ is a pair $\mathfrak{M} = (\mathscr{F}, \upsilon)$ where υ is a function mapping the propositional variables into subsets of F. The inductive definition of "*formula φ is true at point x in model $\mathfrak{M}$*" is the standard one, e.g., the clause for $\Diamond_i$ $(i < n)$ is as follows:

$$\mathfrak{M}, x \models \Diamond_i \psi \ \text{ iff } \ \exists y \ (x R_i y \text{ and } \mathfrak{M}, y \models \psi).$$

Given an n-frame $\mathscr{F}$ and an n-modal formula φ, we say that φ is *satisfiable* in $\mathscr{F}$ if $\mathfrak{M}, x \models \varphi$ for some model $\mathfrak{M}$ on $\mathscr{F}$ and point x in F. Similarly, φ is *valid* in $\mathscr{F}$ if $\mathfrak{M}, x \models \varphi$ for all such $\mathfrak{M}$ and x. $\mathscr{F}$ is a *frame for* a set L of n-modal formulas if all formulas of L are valid in $\mathscr{F}$. L is called a *Kripke complete n-modal logic* if there is some class $\mathscr{C}$ of n-frames such that L is the set of all n-modal formulas which are valid in every member of $\mathscr{C}$. This case we also say that L is the *logic of* $\mathscr{C}$. Well-known Kripke complete unimodal logics are $\mathbf{K}$ (the logic of all 1-frames) and $\mathbf{S5}$ (the logic of all 1-frames (W, R) with R being an equivalence relation on W).

Special n-frames are the following (*n-ary*) *product frames*. Given 1-frames (i.e., usual Kripke frames for unimodal logic) $\mathscr{F}_0 = (W_0, R_0), \dots, \mathscr{F}_{n-1} = (W_{n-1}, R_{n-1})$, their product $\mathscr{F}_0 \times \cdots \times \mathscr{F}_{n-1}$ is defined to be the relational structure

$$(W_0 \times \cdots \times W_{n-1}, \bar{R}_0, \dots, \bar{R}_{n-1})$$

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