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A MAXIMAL BOUNDED FORCING AXIOM

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Abstract. After presenting a general setting in which to look at forcing axioms, we give a hierarchy of generalized bounded forcing axioms that correspond level by level, in consistency strength, with the members of a natural hierarchy of large cardinals below a Mahlo. We give a general construction of models of generalized bounded forcing axioms. Then we consider the bounded forcing axiom for a class of partially ordered sets Γ_1 such that, letting Γ_0 be the class of all stationary-set-preserving partially ordered sets, one can prove the following:

(a) $\Gamma_0 \subseteq \Gamma_1$,
(b) $\Gamma_0 = \Gamma_1$ if and only if NS_{ω_1} is $\aleph_1$-dense.
(c) If $P \notin \Gamma_1$, then $BFA(\{P\})$ fails.

We call the bounded forcing axiom for Γ_1 *Maximal Bounded Forcing Axiom* ($MBFA$). Finally we prove $MBFA$ consistent relative to the consistency of an inaccessible Σ_2-correct cardinal which is a limit of strongly compact cardinals.

§1. A general setting for forcing axioms. Forcing axioms, like the Proper Forcing Axiom (PFA) or the Semiproper Forcing Axiom ($SPFA$), have natural bounded forms, the so–called *bounded forcing axioms*, which were explicitely considered for the first time in [G-S], where the Bounded Proper Forcing Axiom ($BPFA$) is introduced and proved equiconsistent with a small large cardinal hypothesis. Bounded forcing axioms can be characterized as principles of absoluteness under forcing extensions for Σ_1 sentences with parameters in $H(\omega_2)$ (see [B, M]).

We start giving a general definition of unbounded and bounded forcing axiom.

DEFINITION 1.1. Let Γ be a class of partially ordered sets and let κ be a cardinal. $FA(\Gamma)_\kappa$ denotes the following statement:

> Given a partially ordered set $\mathbb{P} \in \Gamma$ and a set $\mathscr{A}$ of size less than κ consisting of maximal antichains of $\mathbb{P}$, there is an $\mathscr{A}$–generic filter $G \subseteq \mathbb{P}$.

DEFINITION 1.2. Let κ and λ be two infinite cardinals and let Γ be a class of partially ordered sets. Then, $BFA(\Gamma)_{\kappa,\lambda}$ denotes the statement obtained in Definition 1.1 by requiring that all $A \in \mathscr{A}$ have size less than λ.

We suggest to make an exception in the above notations when Γ is the class of all ccc partially ordered sets and to use the classical notation in that case, i.e., for every infinite cardinal $\kappa < 2^{\aleph_0}$, to let MA_κ denote $FA(ccc)_{\kappa^+}$ (and, consequently, also $BFA(ccc)_{\kappa^+,\lambda}$ whenever $\lambda \geq \aleph_1$).

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