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SOME WEAK FRAGMENTS OF *HA* AND CERTAIN CLOSURE PROPERTIES

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Abstract. We show that Intuitionistic Open Induction iop is not closed under the rule $DNS(\exists_1^-)$. This is established by constructing a Kripke model of $iop + \neg L_y(2y > x)$, where $L_y(2y > x)$ is universally quantified on x. On the other hand, we prove that iop is equivalent with the intuitionistic theory axiomatized by PA^- plus the scheme of weak $\neg\neg LNP$ for open formulas, where universal quantification on the parameters precedes double negation. We also show that for any open formula $\varphi(y)$ having only y free, $(PA^-)^i \vdash L_y\varphi(y)$. We observe that the theories iop, $i\forall_1$ and $i\Pi_1$ are closed under Friedman's translation by negated formulas and so under VR and IP. We include some remarks on the classical worlds in Kripke models of iop.

§1. Preliminaries. 1.1 Let DOR (resp. PA^-) be the finite set of usual axioms (including Trichotomy) for discretely ordered commutative rings with 1 (resp. their nonnegative parts) in the language $L = \{+, \cdot, <, 0, 1\}$ of arithmetic. Peano Arithmetic PA (resp. Heyting Arithmetic HA) is the classical (resp. intuitionistic, obtained by dropping the principle PEM of excluded middle whose instance PEM_φ on a formula φ is $\varphi \vee \neg\varphi$) first order theory axiomatized by PA^- together with the induction scheme whose instance with respect to a distinguished free variable x on a formula $\varphi(x, \overline{y})$ is

$$I_x\varphi = I_x\varphi(x, \overline{y}) : \forall\overline{y}(\varphi(0, \overline{y}) \wedge \forall x(\varphi(x, \overline{y}) \to \varphi(x+1, \overline{y})) \to \forall x\varphi(x, \overline{y})).$$

1.2 The classical Open Induction fragment Iop of PA is axiomatized by only keeping (besides PA^-) the instances of induction on open, i.e., quantifier-free, formulas. It was first studied by Shepherdson [6]. He constructed a (recursive) nonstandard model proving independence results, such as irrationality of $\sqrt{2}$, from Iop. Let $\tilde{\mathbb{Q}}$ be the field of real algebraic numbers. Shepherdson's model was

$$\begin{aligned}\mathcal{S}_t(\mathbb{N}) &= \cup_{n\in\mathbb{Z}^{>0}}(t^{\frac{1}{n}}\tilde{\mathbb{Q}}[t^{\frac{1}{n}}] + \mathbb{Z})^{\geq 0} \\ &= \{a_m t^{\frac{m}{n}} + a_{m-1}t^{\frac{m-1}{n}} + \cdots + a_1 t^{\frac{1}{n}} + a_0 : n \in \mathbb{Z}^{>0}, m \in \mathbb{N}, a_m, \\ &\qquad \cdots, a_1 \in \tilde{\mathbb{Q}}, a_0 \in \mathbb{Z}, a_m \geq 0, m > 0 \to a_m > 0\}.\end{aligned}$$

This is equipped with the obvious $+$ and $\cdot$ and the (non-Archimedean and consistent with $+$ and $\cdot$) order induced by $t > \mathbb{N}$. We will use Shepherdson's model and also

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