WEAK DISTRIBUTIVITY, A PROBLEM OF VON NEUMANN AND THE MYSTERY OF MEASURABILITY

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Dedicated to Dorothy Maharam Stone

This article investigates the weak distributivity of Boolean σ-algebras satisfying the countable chain condition. It addresses primarily the question when such algebras carry a σ-additive measure. We use as a starting point the problem of John von Neumann stated in 1937 in the Scottish Book. He asked if the countable chain condition and weak distributivity are sufficient for the existence of such a measure.

Subsequent research has shown that the problem has two aspects: one set theoretic and one combinatorial. Recent results provide a complete solution of both the set theoretic and the combinatorial problems. We shall survey the history of von Neumann's Problem and outline the solution of the set theoretic problem. The technique that we describe owes much to the early work of Dorothy Maharam to whom we dedicate this article.

§1. **Complete Boolean algebras and weak distributivity.** A *Boolean algebra* is a set B with Boolean operations $a \vee b$ (join), $a \wedge b$ (meet) and $-a$ (complement), partial ordering $a \leq b$ defined by $a \wedge b = a$ and the smallest and greatest element, $\mathbf{0}$ and $\mathbf{1}$. By Stone's Representation Theorem, every Boolean algebra is isomorphic to an algebra of subsets of some nonempty set S, under operations $a \cup b$, $a \cap b$, $S - a$, ordered by inclusion, with $\mathbf{0} = \emptyset$ and $\mathbf{1} = S$.

If every subset A of B has a least upper bound $\bigvee A$ (and the greatest lower bound $\bigwedge A$) then B is a *complete* Boolean algebra. An *antichain* in B is a set $A \subseteq B$ such that distinct elements $a, b \in A$ are *disjoint* i.e., $a \wedge b = \mathbf{0}$. B satisfies the *countable chain condition* (ccc) if it has no uncountable antichains.

If B is a ccc Boolean σ-algebra, i.e., $\bigvee_{n \in \omega} a_n$ and $\bigwedge_{n \in \omega} a_n$ exist for countable sets, then B is a complete Boolean algebra. For this and other basic facts on Boolean algebras, we refer the reader to [30], [24].