

THE EMPTY SET, THE SINGLETON, AND THE ORDERED PAIR

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Dedicated to the memory of Burton S. Dreben

For the modern set theorist the empty set \emptyset , the singleton $\{a\}$, and the ordered pair $\langle x, y \rangle$ are at the beginning of the systematic, axiomatic development of set theory, both as a field of mathematics and as a unifying framework for ongoing mathematics. These notions are the simplest building blocks in the abstract, generative conception of sets advanced by the initial axiomatization of Ernst Zermelo [1908a] and are quickly assimilated long before the complexities of Power Set, Replacement, and Choice are broached in the formal elaboration of the ‘set of’ $\{\}$ operation. So it is surprising that, while these notions are unproblematic today, they were once sources of considerable concern and confusion among leading pioneers of mathematical logic like Frege, Russell, Dedekind, and Peano. In the development of modern mathematical logic out of the turbulence of 19th century logic, the emergence of the empty set, the singleton, and the ordered pair as clear and elementary set-theoretic concepts serves as a motif that reflects and illuminates larger and more significant developments in mathematical logic: the shift from the intensional to the extensional viewpoint, the development of type distinctions, the logical vs. the iterative conception of set, and the emergence of various concepts and principles as distinctively set-theoretic rather than purely logical. Here there is a loose analogy with Tarski’s recursive definition of truth for formal languages: The mathematical interest lies mainly in the procedure of recursion and the attendant formal semantics in model theory, whereas the philosophical interest lies mainly in the basis of the recursion, truth and meaning at the level of basic predication. Circling back to the beginning, we shall see how central the empty set, the singleton, and the ordered pair were, after all.

§1. The empty set. A first look at the vicissitudes specific to the empty set, or null class, provides an entrée into our main themes, particularly the

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