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# EXTENSIONS OF ORDERED THEORIES BY GENERIC PREDICATES

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**§1. Introduction.** Given a theory $T$ extending that of dense linear orders without endpoints (DLO), in a language $\mathcal{L} \supseteq \{<\}$, we are interested in extensions $T'$ of $T$ in languages extending $\mathcal{L}$ by unary relation symbols that are each interpreted in models of $T'$ as sets that are both dense and codense in the underlying sets of the models.

There is a canonically "wild" example, namely, $T = \mathrm{Th}(\langle \mathbb{R}, <, +, \cdot \rangle)$ and $T' = \mathrm{Th}(\langle \mathbb{R}, <, +, \cdot, \mathbb{Q} \rangle)$. Recall that $T$ is o-minimal, and so every open set definable in any model of $T$ has only finitely many definably connected components. But it is well known that $\langle \mathbb{R}, <, +, \cdot, \mathbb{Q} \rangle$ defines every real Borel set, in particular, every open subset of any finite cartesian power of $\mathbb{R}$ and every subset of any finite cartesian power of $\mathbb{Q}$. To put this another way, the definable open sets in models of $T$ are essentially as simple as possible, while $T'$ has a model where the definable open sets are as complicated as possible, as is the structure induced on the new predicate.

In contrast to the preceding example, if $\mathbb{R}_{\mathrm{alg}}$ is the set of real algebraic numbers and $T' = \mathrm{Th}(\langle \mathbb{R}, <, +, \cdot, \mathbb{R}_{\mathrm{alg}} \rangle)$, then no model of $T'$ defines any open set (of any arity) that is not definable in the underlying model of $T$. More generally, if $\mathfrak{B}$ is an o-minimal expansion of a densely ordered group and $A$ is the underlying set of a dense elementary substructure of $\mathfrak{B}$, then $\mathrm{Th}(\langle \mathfrak{B}, A \rangle)$ is rather well behaved with respect to $\mathrm{Th}(\mathfrak{B})$, in particular, every open set definable in $\langle \mathfrak{B}, A \rangle$ is definable in $\mathfrak{B}$; see [6, Section 5] for details. There is an orthogonal complement [7]: If $E \subseteq B$ is dense and definably independent with respect to $\mathfrak{B}$, then again, every open set definable in $\langle \mathfrak{B}, E \rangle$ is definable in $\mathfrak{B}$.

Another class of examples is treated in [6, Section 6], namely, extensions $T'$ of o-minimal theories $T$ by "generic (unary) predicates"; this material was included in [6] only to illustrate some of the broader themes of that paper as a whole. Here, we shall relax the assumption that $T$ be o-minimal and consider such extensions $T'$ in their own right. Some preliminary discussion of the underlying intuitive ideas is in order.

Fix for the moment a positive integer $N$. We want to run a fair "pick $N$" lottery game on balls colored either black or white. The ways that we can mix and draw

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