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DIAGONAL PRIKRY EXTENSIONS

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§1. Introduction. It is a well-known phenomenon in set theory that problems in infinite combinatorics involving singular cardinals and their successors tend to be harder than the parallel problems for regular cardinals. Examples include the behaviour of cardinal exponentiation, the extent of the tree property, the extent of stationary reflection, and the existence of non-free almost-free abelian groups. The explanation for this phenomenon lies in inner model theory, in particular *core models* and *covering lemmas*. If W is an inner model of V then

1. *W strongly covers V* if every uncountable set of ordinals is covered by a set of the same V-cardinality lying in W.
2. *W weakly covers V* if W computes the successor of every V-singular cardinal correctly.

Strong covering implies weak covering.

In inner model theory there are many theorems of the general form "if there is no inner model of large cardinal hypothesis X then there is an L-like inner model K_X which Y covers V". Here the L-like properties of K_X always include GCH and Global Square. Examples include

1. X is "$0^\sharp$ exists", K_X is L, Y is "strongly".
2. X is "there is a measurable cardinal", K_X is the Dodd-Jensen core model, Y is "strongly".
3. X is "there is a Woodin cardinal", K_X is the core model for a Woodin cardinal, Y is "weakly".

If V is strongly covered by an inner model with GCH then the SCH holds in V. If V is weakly covered by a model with Global Square then $\Box_\kappa$ holds in V for every V-singular cardinal κ, and this also exerts a strong influence on the combinatorics of κ and κ^+; for example there is a special κ^+-Aronszajn tree. and there is a non-reflecting stationary set in κ^+.

Research on problems involving singular cardinals has given birth to the field of *singular cardinal combinatorics*. For the reasons we have discussed, the combinatorics of singular cardinals is closely bound up with large cardinals and L-like combinatorial principles, and involves many questions of consistency and independence. This is by no means the whole story: working in ZFC set theory Shelah has

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