THE JOURNAL OF SYMBOLIC LOGIC
Volume 74, Number 4, Dec. 2009

FORBIDDEN INTERVALS

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§1. Introduction. Many classical statements of set theory are settled by the existence of generic elementary embeddings that are analogous the elementary embeddings posited by large cardinals. [2] The embeddings analogous to measurable cardinals are determined by uniform, κ-complete precipitous ideals on cardinals κ. Stronger embeddings, analogous to those originating from supercompact or huge cardinals are encoded by normal fine ideals on sets such as $[\kappa]^{<\lambda}$ or $[\kappa]^{\lambda}$.

The embeddings generated from these ideals are limited in ways analogous to conventional large cardinals. Explicitly, if $j : V \rightarrow M$ is a generic elementary embedding with critical point κ and $\lambda = \sup_{n\in\omega} j^n(\kappa)$ and the forcing yielding j is λ-saturated then $j``\lambda^+ \notin M$. (See [2].)

Ideals that yield embeddings that are analogous to strongly compact cardinals have more puzzling behavior and the analogy is not as straightforward. Some natural ideal properties of this kind have been shown to be inconsistent:

THEOREM 1 (Kunen). *There is no ω_2-saturated, countably complete uniform ideal on any cardinal in the interval $[\aleph_\omega, \aleph_{\omega_1})$.*

Generic embeddings that arise from countably complete, ω_2-saturated ideals have the property that $\sup j^n(\omega_1) = \aleph_{\omega^V}$. So the Kunen result is striking in that it apparently allows strong ideals to exist above the conventional large cardinal limitations. The main result of this paper is that it is consistent (relative to a huge cardinal) that such ideals exist.

1.1. More forbidden intervals. We can generalize Kunen's theorem as follows:[1]

DEFINITION 2. *Let κ be a regular cardinal. Define $\mathscr{C}_\kappa$ to be the smallest class of ordinals such that*:

1. $\kappa \subset \mathscr{C}_\kappa$.
2. *If $\alpha, \beta \in \mathscr{C}$, and $\aleph_\alpha(\beta) \geq \aleph_\omega(\kappa)$ then $\aleph_\alpha(\beta) \in \mathscr{C}_\kappa$.*
3. *If $\beta \in \mathscr{C}_\kappa$ and $\beta \geq \kappa$, then every cardinal in the $[\beta, \aleph_\kappa(\beta))$ belongs to $\mathscr{C}_\kappa$.*

An easy induction shows that for all $\kappa, [\kappa, \aleph_\omega(\kappa)) \cap \mathscr{C}_\kappa = \emptyset$. The next result generalizes Kunen's theorem:

PROPOSITION 3. *Let κ be a successor cardinal. Suppose that λ is a regular cardinal in $\mathscr{C}_\kappa$. Then there is no κ-complete, κ^+-saturated uniform ideal on λ.*

Received October 7, 2005.
The author would like to acknowledge research support from NSF grant DMS 0701030.

[1] We define $\aleph_\delta(\kappa)$ to be the δ^{th} cardinal successor of κ

1943-5886/09/7404-0001/$2.90