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ZERO-ONE LAW AND DEFINABILITY OF LINEAR ORDER

HANNU NIEMISTÖ

§1. Introduction. A logic $\mathscr{L}$ has a limit law, if the asymptotic probability of every query definable in $\mathscr{L}$ converges. It has a 0–1-law if the probability converges to 0 or 1. The 0–1-law for first-order logic on relational vocabularies was independently found by Glebski et al. [6] and Fagin [5]. Later it has been shown for many other logics, for instance for fragments of second order logic [12], for finite variable logic [13] and for FO extended with the rigidity quantifier [3]. Lynch [14] has shown a limit law for first-order logic on vocabularies with unary functions.

We say that two formulas or two logics are almost everywhere equivalent, if they are equivalent on a class of structures whose asymptotic probability measure is one [7]. A 0–1-law is usually proved by showing that every quantifier of the logic has almost everywhere quantifier elimination, i.e., every formula with just one quantifier in front of it is almost everywhere equivalent to a quantifier-free formula. Besides proving 0–1-law, this implies that the logic is (weakly) almost everywhere equivalent to first-order logic.

The aim of this paper is to study, whether a logic with a 0–1-law can have greater expressive power than FO in the almost everywhere sense and to what extent. In particular, we are interested on the definability of linear order. Because a 0–1-law determines the almost everywhere expressive power of the sentences of the logic completely, but does not say anything about formulas explicitly, we have to assume some regularity on logics. We will therefore mostly consider extensions of first-order logic with generalized quantifiers.

The paper consists of two results. First we show that a logic capable of defining a linear order on almost all graphs cannot have a limit law. This is done by strengthening a result by Compton, Henson and Shelah [2]. Similar ideas are used also in [11]. We show that FO does not have a limit law on ordered random graphs even if the order can be chosen freely after the edge relation is determined. As a by-product, we demonstrate how one obvious approach to proving a 0–1 law for order-invariant first-order logic fails.

On the other hand, we construct a quantifier Q such that FO(Q) has a 0–1-law on all vocabularies, but it can define a non-trivial set on almost all graphs, thus FO(Q) is not almost everywhere equivalent to FO. We use the concept of random logic in the construction and show the 0–1-law for the logic by extending some ideas in [3]. In addition, we show that IFP(Q) defines a linear order on almost every graph. This proof is inspired by [10] and its adaptation to a logical form in [7].

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