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HOW ENUMERATION REDUCIBILITY YIELDS EXTENDED HARRINGTON NON-SPLITTING

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§1. Introduction. Sacks [16] showed that every computably enumerable (c.e.) degree $> \boldsymbol{0}$ has a c.e. splitting. Hence, relativising, every c.e. degree has a Δ_2 splitting above each proper predecessor (by 'splitting' we understand 'nontrivial splitting'). Arslanov [1] showed that $\boldsymbol{0}'$ has a d.c.e. splitting above each c.e. $\boldsymbol{a} < \boldsymbol{0}'$. On the other hand, Lachlan [11] proved the existence of a c.e. $\boldsymbol{a} > \boldsymbol{0}$ which has no c.e. splitting above some proper c.e. predecessor, and Harrington [10] showed that one could take $\boldsymbol{a} = \boldsymbol{0}'$. Splitting and nonsplitting techniques have had a number of consequences for definability and elementary equivalence in the degrees below $\boldsymbol{0}'$.

Heterogeneous splittings are best considered in the context of cupping and noncupping. Posner and Robinson [15] showed that every nonzero Δ_2 degree can be nontrivially cupped to $\boldsymbol{0}'$, and Arslanov [1] showed that every c.e. degree $> \boldsymbol{0}$ can be d.c.e. cupped to $\boldsymbol{0}'$ (and hence since every d.c.e., or even n-c.e., degree has a nonzero c.e. predecessor, every n-c.e. degree $> \boldsymbol{0}$ is d.c.e. cuppable). Cooper [4] and Yates (see Miller [13]) showed the existence of degrees noncuppable in the c.e. degrees. Moreover, the search for relative cupping results was drastically limited by Cooper [5], and Slaman and Steel [17] (see also Downey [9]), who showed that there is a nonzero c.e. degree $\boldsymbol{a}$ below which even Δ_2 cupping of c.e. degrees fails.

We prove below what appears to be the strongest possible of such nonsplitting and noncupping results.

THEOREM 1.1. *There exists a computably enumerable degree $\boldsymbol{a} < \boldsymbol{0}'$ such that there exists no nontrivial cuppings of c.e. degrees above $\boldsymbol{a}$ in the Δ_2 degrees above $\boldsymbol{a}$.*

In fact, if we consider the extended structure of the enumeration degrees, Theorem 1.1 is a corollary of the even stronger result:

THEOREM 1.2. *There exists a Π_1 e-degree $\boldsymbol{a} < \boldsymbol{0}'_e$ such that there exist no nontrivial cuppings of Π_1 e-degrees above $\boldsymbol{a}$ in the Σ_2 e-degrees above $\boldsymbol{a}$.*

This would appear to be the first example of a structural feature of the Turing degrees obtained via a proof in the wider context of the enumeration degrees (rather than the other way round).

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