ON THE CONSISTENCY STRENGTH OF THE INNER MODEL HYPOTHESIS

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The *Inner Model Hypothesis* (IMH) and the *Strong Inner Model Hypothesis* (SIMH) were introduced in [4]. In this article we establish some upper and lower bounds for their consistency strength.

We repeat the statement of the IMH, as presented in [4]. A sentence in the language of set theory is *internally consistent* iff it holds in some (not necessarily proper) inner model. The meaning of internal consistency depends on what inner models exist: If we enlarge the universe, it is possible that more statements become internally consistent. The *Inner Model Hypothesis* asserts that the universe has been maximised with respect to internal consistency:

The Inner Model Hypothesis (IMH): If a statement φ without parameters holds in an inner model of some outer model of V (i.e., in some model compatible with V), then it already holds in some inner model of V.

Equivalently: If φ is internally consistent in some outer model of V then it is already internally consistent in V. This is formalised as follows. Regard V as a countable model of Gödel-Bernays class theory, endowed with countably many sets and classes. Suppose that V^* is another such model, with the same ordinals as V. Then V^* is an *outer model of V* (V is an *inner model of V^**) iff the sets of V^* include the sets of V and the classes of V^* include the classes of V. V^* is *compatible with V* iff V and V^* have a common outer model.

REMARK. The Inner Model Hypothesis, like Lévy-Shoenfield absoluteness, is a form of absoluteness between V and arbitrary outer models of V, which need not be generic extensions of V. Formally speaking, the notion of "arbitrary outer model" does depend on the background universe in which V is situated as a countable model. However, a typical model of the IMH is minimal in the sense that for some real R, it is the smallest transitive model of Gödel-Bernays containing R (see Theorem 8 below). For minimal models, the choice of background universe is irrelevant, and if there is a model of the IMH then there is a minimal one. Thus we may in fact regard the IMH as an intrinsic hypothesis about V, independent of any background universe. An alternative way to "internalise" the IMH is to restrict the notion of outer model to class-generic extensions which preserve the axioms